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Dobrushin Uniqueness Theorem and Logarithmic Sobolev Inequalities

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We formulate a condition on a local specification $\mathscr{E}$ on a countable product space M^{Γ}, M being a Riemannian manifold or a discrete set $\{-1, +1\}$, assuring that the corresponding set of Gibbs measures consists of a unique measure μ satisfying a logarithmic Sobolev inequality.

0. INTRODUCTION

Let M be a finite dimensional, connected smooth Riemannian manifold and let $(\,|\,)$, resp. ∂, be the associated inner product, resp. gradient operator. By $\mathscr{M}$ we denote the Borel σ-algebra of sets in M.

Let $\Omega \equiv M^{\Gamma}$ be a countable product space and let Σ denote the σ-algebra of its subsets generated by product topology. In the defined space we have a family of natural projections

$$\Omega \ni \omega \mapsto \omega_{\Lambda} \in M^{\Lambda}, \qquad \Lambda \subset \Gamma. \tag{0.1}$$

(If $\Lambda = \{i\}$ for some $i \in \Gamma$ we will write simply ω_i instead of $\omega_{\{i\}}$.) We can consider Ω as the product $M^{\Lambda} \times M^{\Lambda^c}$, with $\Lambda^c \equiv \Gamma \backslash \Lambda$ for any nonempty set $\Lambda \subset \Gamma$. If $\Lambda \subset \Gamma$ then we set Σ_{Λ} to be the smallest sub-σ-algebra of Σ, such that all projections ω_i, $i \in \Lambda$, are Σ_{Λ} measurable. For $\Lambda \equiv \Gamma$ we have $\Sigma_{\Gamma} \equiv \Sigma$. The σ-algebra Σ_{∞} of events at infinity is defined

$$\Sigma_{\infty} = \bigcap_{\Lambda \in \mathscr{F}} \Sigma_{\Lambda^c}, \tag{0.2}$$

where intersection goes over all elements of the family $\mathscr{F}$ of finite sets in Γ. If a real function f on Ω is Σ_{Λ} measurable we write simply $f \in \Sigma_{\Lambda}$. By

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$|f|$ we denote the absolute value of a function f. It is useful to define embeddings

$$\delta_\Lambda^{\tilde{\omega}}: M^\Lambda \to \Omega \tag{0.3}$$

with $\tilde{\omega} \in \Omega$ and $\Lambda \subset \mathbf{\Gamma}$, so that for any measurable real function f on (Ω, Σ) we can define a function

$$\delta_\Lambda^{\tilde{\omega}} f: M^\Lambda \to \mathbf{R} \tag{0.4}$$

by

$$(\delta_\Lambda^{\tilde{\omega}} f)(\omega_\Lambda) := f(\omega_\Lambda \times \tilde{\omega}_{\Lambda^c}). \tag{0.5}$$

Let $\mathscr{C}^1(\Omega)$ (resp. $\mathscr{C}_b^1(\Omega)$) be the space of functions $f \in \Sigma$ for which $\delta_i^{\tilde{\omega}} f \in \mathscr{C}^1(M)$ (resp. $\in \mathscr{C}_b^1(M)$) for all $i \in \mathbf{\Gamma}$ and $\tilde{\omega} \in \Omega$, where $\mathscr{C}^1(M)$ (resp. $\mathscr{C}_b^1(M)$) denotes the space of differentiable real functions on M (resp. additionally with bounded derivatives). For $f \in \mathscr{C}^1(\Omega)$ we define a gradient

$$\nabla f \equiv (\nabla_i f)_{i \in \mathbf{\Gamma}} \tag{0.6}$$

by

$$(\nabla_i f)(\omega) := \partial(\delta_i^\omega f)(\omega_i) \tag{0.7}$$

and we set

$$|\nabla f|^2 \equiv \sum_{i \in \mathbf{\Gamma}} (\nabla_i f \mid \nabla_i f). \tag{0.8}$$

Let μ be a probability measure on (Ω, Σ). We write $\mu(f)$, or simply μf, to denote the expectation value of a measurable real function f computed with the measure μ. By $\mathscr{H}_+(\mu)$ we denote the space of functions $f \in \mathscr{C}^1(\Omega)$ for which $\mu\,|\nabla f|^2$ and μf^2 are finite. For two measurable functions f and g on (Ω, Σ) we define the truncated correlation function

$$\mu(f, g) := \mu fg - \mu f \mu g. \tag{0.9}$$

A probability measure μ satisfies the *logarithmic Sobolev inequality* (for short, log-S) with a coefficient $0 < c < \infty$ iff

$$\mu f^2 \log |f| \leqslant c\mu\,|\nabla f|^2 + \mu f^2 \log(\mu f^2)^{1/2} \tag{0.10}$$

for any function $f \in \mathscr{H}_+(\mu)$. The logarithmic Sobolev inequalities have been introduced in [1] as a generalization of classical Sobolev inequalities to infinite dimensional spaces. It was shown there that the log-Sobolev inequality is equivalent to the hypercontractivity of the semigroup

$P_t \equiv e^{-tH}$, $t > 0$, with generator H defined by the Dirichlet form $\mu\,|\nabla f|^2$ (if closable). It was also observed there that the log-S inequalities have a remarkable inductive property, which can be formulated as follows: If log-S holds for a probability measure μ on $(M, \mathcal{M})^n$ with coefficient c_μ and for a probability measure ρ on $(M, \mathcal{M})$ with coefficient c_ρ, then the probability measure $\mu \otimes \rho$ on $(M, \mathcal{M})^{n+1}$ satisfies log-S with coefficient $\max(c_\mu, c_\rho)$. This in particular implies that an infinite product $\rho^{\otimes \Gamma}$ satisfies log-S with coefficient c_ρ. Using this one may conclude (see [1]) that also a Gaussian measure on the space of tempered distributions $(\mathscr{S}', \mathscr{B})$, representing the free euclidean field [33], satisfies log-S. To find a first nontrivial example of probability measure on an infinite dimensional space, i.e., a nonproduct and non-Gaussian measure, was possible after the invention by Bakry and Emery [2] of a very nice sufficient condition, called the Γ_2-criterion, implying log-S. Using the Γ_2-criterion it was shown in [3] that the infinite volume measures of statistical mechanical systems on $(S^N)^\Gamma$, S^N being an $N \geqslant 2$ dimensional unit sphere, at sufficiently high temperatures satisfy log-S. An extension of these results along the same lines has been worked out in a recent work [4]. The purpose of our work is to give another condition assuring a probability measure μ on an infinite dimensional space satisfies log-S. The central role in our approach is played by a Gibbs structure [5, 6]. We consider a loca specification $\mathscr{E} = \{E^{\cdot}_\Lambda\}_{\Lambda \in \mathscr{F}}$ whose elements are the probability kernels satisfying the following requirements:

(i) For any $\Lambda \in \mathscr{F}$ and $\omega \in \Omega$, E^ω_Λ is a probability measure on (Ω, Σ).

(ii) For any bounded function $f \in \Sigma$ and $\Lambda \in \mathscr{F}$, the function

$$\Omega \ni \omega \mapsto E^\omega_\Lambda f$$

is Σ_{Λ^c} measurable and for all $f \in \Sigma_{\Lambda^c}$ we have

$$E^\omega_\Lambda f = f(\omega).$$

(iii) (Compatibility Condition) For any $\Lambda, \Lambda' \in \mathscr{F}$, $\Lambda' \subset \Lambda$ we have

$$E^\omega_\Lambda E^{\cdot}_{\Lambda'} = E^\omega_\Lambda.$$

For a local specification $\mathscr{E}$ we define the set $\mathscr{G}(\mathscr{E})$ of associated Gibbs measures to be the set of probability measures μ on (Ω, Σ) fulfilling

$$\mu E^{\cdot}_\Lambda = \mu \tag{0.11}$$

for all $\Lambda \in \mathscr{F}$. By $\partial\mathscr{G}(\mathscr{E})$ we denote the set of extremal Gibbs measures, i.e., of measures $\mu \in \mathscr{G}(\mathscr{E})$ which have no nontrivial convex linear representation in terms of other elements of $\mathscr{G}(\mathscr{E})$.

We will assume the family $\mathscr{E} = \{E^{\cdot}_{\Lambda}\}_{\Lambda \in \mathscr{F}}$ to be a differentiable specification (or for short $\mathscr{C}^1$-specification) in the sense that for any function $f \in \mathscr{C}^1(\Omega)$ the functions $E^{\cdot}_{\Lambda} f$, $\Lambda \in \mathscr{F}$ are also in $\mathscr{C}^1(\Omega)$.

Our criterion for a Gibbs measure μ to satisfy log-S is formulated, similarly as the Dobrushin uniqueness condition [7–9], in terms of one point kernels $E^{\cdot}_i \in \mathscr{E}$, $i \in \Gamma$. We assume the following *condition* (C):

(Ci) The probability measures E_i^{ω} satisfy log-S with a constant $0 < c_0 < \infty$ independent of $i \in \Gamma$ and $\omega \in \Omega$.

(Cii) There is a matrix $\mathbf{c} \equiv \{C_{ij} \geqslant 0\}_{i,j \in \Gamma}$ such that for any strictly positive function $f \in \mathscr{C}^1(\Omega)$

$$|\nabla_j (E_i^{\omega} f^2)^{1/2}| \leqslant (E_i^{\omega} |\nabla_j f|^2)^{1/2} + C_{ji} (E_i^{\omega} |\nabla_i f|^2)^{1/2} \tag{0.12}$$

for any $i, j \in \Gamma$, $i \neq j$. For $i = j$ the lhs of (0.12) equals zero by definition of local specification and therefore we have $C_{ii} \equiv 0$.

The inequality (0.12) has formally the same form as in Lemma C2.2 inequality (8) of [8], essentially used there for the proof of the Dobrushin uniqueness theorem.

If the matrix $\mathbf{c}$ is "sufficiently small" the above condition (C) allows us to get an inductive proof of the log-Sobolev inequality. This generalizes the inductive character of log-S mentioned before for product measures.

In Sections 1 and 2 of our work we prove the following result:

THEOREM 0.1. *Let $\mathscr{E} = \{E^{\cdot}_{\Lambda}\}_{\Lambda \in \mathscr{F}}$ be a $\mathscr{C}^1$ local specification satisfying condition* (C) *with a corresponding matrix $\mathbf{c} \equiv \{C_{ij}\}$. Suppose*

$$\gamma \equiv \sup_{i \in \Gamma} \max \left(\sum_{j \in \Gamma} C_{ji}, \sum_{j \in \Gamma} C_{ij} \right) < 1. \tag{0.13}$$

Then the unique Gibbs measure $\mu \in \mathscr{G}(\mathscr{E})$ satisfies the log-Sobolev inequality

$$\mu(f^2 \log |f|) \leqslant c\mu\, |\nabla f|^2 + \mu f^2 \log(\mu f^2)^{1/2} \tag{0.14}$$

with a constant

$$0 < c \leqslant c_0 (1 - \gamma)^{-2} \tag{0.15}$$

for any $f \in \mathscr{H}_+(\mu)$.

In Section 3 we give examples of applications of Theorem 0.1 to the cases important in statistical mechanics and quantum field theory when the manifold M can be compact as well as noncompact. Section 4 is devoted to a generalization of the preceding result to the case of discrete spin systems when $M \equiv \{-1, +1\}$. Then the role of gradient operator ∂ is played by the projection operator onto nonconstant functions. In this situation we shall have to consider a modified condition (Cii) with the first term on the rhs of (0.12) multiplied by a constant $1 \leqslant \alpha < \infty$. Under some smallness condition on γ we get a similar result, with the corresponding constant in the log-Sobolev inequality dependent now also on α. This generalization is important for application to investigation of stochastic dynamics of discrete spin systems considered in statistical mechanics (see [12–15, 18, 19] and references therein). In the last section we give some discussion of our results. We argue there that log-Sobolev inequalities should hold far outside of Dobrushin's uniqueness region.

We propose also to consider some problems connected with stochastic dynamics, which we think are interesting both for mathematics and physics.

1. Logarithmic Sobolev Inequalities for Gibbs Measures

This section is devoted to proving the log-Sobolev inequality for a Gibbs measure $\mu \in \mathscr{G}(\mathscr{E})$, i.e., the inequality

$$\mu f^2 \log |f| \leqslant c\mu \, |\nabla f|^2 + \mu f^2 \log(\mu f^2)^{1/2}, \tag{1.1}$$

assuming $\mathscr{E} = \{E^{\cdot}_{\Lambda}\}_{\Lambda \in \mathscr{F}}$ is a differentiable local specification fulfilling condition (C) with a corresponding matrix $\mathbf{c}$ satisfying (0.13). Let us mention that according to the results of [1] it is sufficient to prove (1.1) only for strictly positive functions f from a suitable dense subset of $\mathscr{H}_+(\mu)$. Our strategy is the following: Let f be a bounded positive function in $\mathscr{C}_b^1(\Omega)$ and $f \in \Sigma_{\tilde{\Lambda}}$ for some $\tilde{\Lambda} \in \mathscr{F}$. We begin by applying the definition of Gibbs measure (0.11) with a kernel $E^{\cdot}_{i_1}$, $i_1 \in \Gamma$, and the uniform log-Sobolev inequality for this kernel (condition (Ci)).

$$\begin{aligned}
\mu f^2 \log |f| &= \mu(E_{i_1} f^2 \log |f|) \\
&\leqslant \mu[c_0 E_{i_1} |\nabla_{i_1} f|^2 + E_{i_1} f^2 \log(E_{i_1} f^2)^{1/2}] \\
&= c_0 \mu \, |\nabla_{i_1} f|^2 + \mu(E_{i_1} f^2 \log(E_{i_1} f^2)^{1/2}).
\end{aligned} \tag{1.2}$$

Next we take another point $i_2 \in \Gamma$ and apply the same arguments to the second term on the rhs of (1.2). This together with (1.2) gives the inequality

$$\begin{aligned}
\mu f^2 \log |f| \leqslant{}& c_0(\mu \, |\nabla_{i_1} f|^2 + \mu \, |\nabla_{i_2}(E_{i_1} f^2)^{1/2}|^2) \\
&+ \mu(E_{i_2} E_{i_1} f^2 \log(E_{i_2} E_{i_1} f^2)^{1/2}).
\end{aligned} \tag{1.3}$$

We will take a sequence of points $\{i_n \in \Gamma\}_{n \in \mathbf{N}}$ and apply iteratively the above arguments. In this way after the nth, $n \geqslant 2$, step, we obtain

$$\mu f^2 \log |f| \leqslant c_0 \left(\mu |\nabla_{i_1} f|^2 + \sum_{k=1}^{n-1} \mu |\nabla_{i_{k+1}} (E_{i_k} \cdots E_{i_1} f^2)^{1/2}|^2 \right)$$
$$+ \mu((E_{i_n} \cdots E_{i_1} f^2) \log(E_{i_n} \cdots E_{i_1} f^2)^{1/2}). \tag{1.4}$$

Our goal will be to show the following

LEMMA 1.1. *There is a sequence* $\mathbf{I} \equiv \{i_k \in \mathbf{\Gamma}\}_{k \in \mathbf{N}}$ *such that assumption* (0.13) *for the matrix condition* (Cii) *implies the bound*

$$\mu |\nabla_{i_1} f|^2 + \sum_{k=1}^{n-1} \mu |\nabla_{i_{k+1}} (E_{i_k} \cdots E_{i_1} f^2)^{1/2}|^2 \leqslant (1-\gamma)^{-2} \mu |\nabla f|^2 \tag{1.5}$$

for any $n \in \mathbf{N}$ *and a positive function* $f \in \mathscr{H}_+(\mu)$.

Under the same assumption and with the same sequence $\mathbf{I} \equiv \{i_k \in \Gamma\}_{k \in \mathbf{N}}$, similar considerations as the one used to show the bound (1.5) will give also

LEMMA 1.2. *For any bounded positive function* $f \in \mathscr{C}_b^1(\Omega)$, $f \in \Sigma_{\tilde{\Lambda}}$, *with* $\tilde{\Lambda} \in \mathscr{F}$

$$\lim_{n \to \infty} E_{i_n} \cdots E_{i_1} f^2 = \mu f^2 \tag{1.6}$$

the limit being understood in $L_1(\mu)$ *and in the case of a compact manifold M also in the supremum norm,*

The last lemma implies that the second term on the rhs of (1.4) converges to $\mu f^2 \log(\mu f^2)^{1/2}$. Combining this, (1.5), and (1.4) we get the log-Sobolev inequality for the Gibbs measure μ with corresponding coefficient

$$c \leqslant c_0(1-\gamma)^{-2}. \tag{1.7}$$

This ends the proof of (1.1). In the course of the proof of Lemma 1.2 we show also that the measure μ is unique.

In view of (1.6), representing a Gibbs measure μ as an infinite "convolution" of probability measures satisfying log-S (with uniformly bounded coefficients), it is no wonder that also the measure μ satisfies the log-Sobolev inequality. This generalizes the inductive property of log-S observed by L. Gross [1]. Let us note that the representation (1.6) is a general feature of Dobrushin uniqueness theory [7–11].

Proof of Lemma 1.1. At the beginning we give a construction of some (natural) sequence $\mathbf{I} \equiv \{i_k \in \mathbf{\Gamma}\}_{k \in \mathbf{N}}$ going infinitely many times through each point of the lattice in the sense that for any $i \in \mathbf{\Gamma}$ it contains an infinite subsequence $\{i_{k_n} : i_{k_n} = i\}_{n \in \mathbf{N}}$. (Let us note however that essentially our arguments are independent of a specific sequence going infinitely many times through each point of the lattice.)

We take an increasing sequence $\mathscr{F}_0 \equiv \{\Lambda_l \in \mathscr{F}\}_{l \in \mathbf{N}}$, called a countable base of $\mathbf{\Gamma}$, defined by the condition that

$$\forall \Lambda \in \mathscr{F}\ \exists l_0 \in \mathbf{N}, \qquad \Lambda \subset \Lambda_{l_0}. \tag{1.8}$$

For $m \in \mathbf{N}$ we introduce the numbers

$$K_m \equiv \sum_{l=1}^{m} |\Lambda_l| \tag{1.9}$$

with $|\Lambda|$ denoting the number of points in a set $\Lambda \in \mathscr{F}$. We fix also an order $\preccurlyeq$ in $\mathbf{\Gamma}$ satisfying

$$\forall i \in \Lambda_l,\ j \in \Lambda_{l+1} \backslash \Lambda_l, \qquad i \prec j \tag{1.10}$$

for any $l \in \mathbf{N}$.

We define the sequence $\mathbf{I} \equiv \{i_k \in \mathbf{\Gamma}\}_{k \in \mathbf{N}}$ inductively as follows: For $k \in [1, K_1]$ we take $i_k \in \Lambda_1$ ordered so that $i_k \prec i_{k+1}$ and for $k \in [K_m, K_{m+1}]$, $m \in \mathbf{N}$, we take all points $i_k \in \Lambda_{m+1}$ ordered according to the order $\preccurlyeq$.

Now for $f \in \mathscr{H}_+(\mu)$ we would like to consider a particular term from the lhs of (1.5) with $k \geqslant 1$. Let us introduce the following notation

$$F_0 \equiv f \tag{1.11}$$

and for $k \in \mathbf{N}$

$$F_k \equiv (E_{i_k} F_{k-1}^2)^{1/2} = (E_{i_k} \cdots E_{i_1} f^2)^{1/2}. \tag{1.12}$$

We shall analyze the expressions

$$|\nabla_{i_{k+1}} F_k| = |\nabla_{i_{k+1}} (E_{i_k} \cdots E_{i_1} f^2)^{1/2}|. \tag{1.13}$$

Applying condition (Cii) we get

$$|\nabla_{i_{k+1}} F_k| \leqslant (E_{i_k} |\nabla_{i_{k+1}} F_{k-1}|^2)^{1/2} + C_{i_{k+1} i_k} (E_{i_k} |\nabla_{i_k} F_{k-1}|^2)^{1/2}. \tag{1.14}$$

Iteration of this step yields the following inequality

$$|\nabla_{i_{k+1}} (E_{i_k} \cdots E_{i_1} f^2)^{1/2}| \leqslant \sum_{i \in \mathbf{\Gamma}} \lambda_{i_{k+1} i}^{(k+1)} (E_{i_k} \cdots E_{i_1} |\nabla_i f|^2)^{1/2} \tag{1.15}$$

with some matrix

$$\lambda^{(k+1)} \equiv \{\lambda_{ji}^{(k+1)}\} \tag{1.16}$$

satisfying the (crude) bound

$$0 \leqslant \lambda_{ji}^{(k+1)} \leqslant \delta_{ji} + \sum_{n=1}^{k} (C^n)_{ji}. \tag{1.17}$$

By squaring (1.15) and using the Schwarz inequality we get

$$\begin{aligned} &|\nabla_{i_{k+1}}(E_{i_k} \cdots E_{i_1} f^2)^{1/2}|^2 \\ &\leqslant \left(\sum_{j \in \Gamma} \lambda_{i_{k+1} j}^{(k+1)} \right) \sum_{i \in \Gamma} \lambda_{i_{k+1} i}^{(k+1)} E_{i_k} \cdots E_{i_1} |\nabla_i f|^2. \end{aligned} \tag{1.18}$$

Let us now observe that from the bound (1.17) and condition (0.13) we have the upper bound

$$\sum_{j \in \Gamma} \lambda_{i_{k+1} j}^{(k+1)} < (1-\gamma)^{-1}. \tag{1.19}$$

Using this and taking the expectation of both sides of (1.18) we get

$$\mu\, |\nabla_{i_{k+1}}(E_{i_k} \cdots E_{i_1} f^2)^{1/2}|^2 \leqslant (1-\gamma)^{-1} \sum_{i \in \Gamma} \lambda_{i_{k+1} i}^{(k+1)} \mu\, |\nabla_i f|^2, \tag{1.20}$$

where in the last step on the right hand side we used the definition of Gibbs measure (and for $k=0$, what corresponds to the inequality for the first term from the lhs of (1.5), we set $\lambda_{i_1 i}^{(1)} \equiv \delta_{i_1 i}$). Now we sum inequalities (1.20) over $k \geqslant 0$. The sum of left hand sides of (1.20) is exactly the left hand side of (1.5) in Lemma 1.1. To discuss the sum of the right hand sides of (1.20) let us analyze in more detail the matrices $\lambda^{(k+1)}$ coming from the induction (1.14). First of all from the definition of the matrix $\lambda^{(k+1)}$ (obtained by iteration of (1.14)) we see that $\lambda_{ji}^{(k+1)} \neq 0$ only for $j = i_{k+1}$ and $i \in \{i_1, ..., i_{k+1}\}$. The matrix elements $\lambda_{ji}^{(k+1)}$ are defined by the paths $\{j_1, ..., j_l\}$ with at most k steps (i.e., $1 \leqslant l \leqslant k+1$) and $j_1 \equiv i_{k+1}, j_l = i$. Because $C_{ii} \equiv 0$, we see that if $l > 1$ we have $j_m \neq j_{m+1}$ and then with the bond (j_m, j_{m+1}) we have associated a factor $C_{j_m j_{m+1}}$. Let us note that the path (j_1, j_1), with one step and identical initial and final points $j_1 = i_{k+1}$, can appear if and only if $i_{k+1} \notin \{i_1, ..., i_k\}$. If we fix $j \in \Gamma$ and look for $k \in \mathbf{Z}^+$ such that $i_{k+1} = j$, we see that we can get the term δ_{ji} in $\lambda_{ji}^{(k+1)}$ only for the first $k = k_0 \in \mathbf{Z}^+$ such that $i_{k_0+1} = j$. Let us see also what we get when i_{k+1} appears at least once in the sequence $\{i_1, ..., i_k\}$. Let $1 \leqslant l \leqslant k$ be the biggest number such that $i_l = i_{k+1}$. Then using (Cii) and the triangle inequality we can continue (1.14) by expanding (with use of (Cii) the first term on its rhs. This gives

$$\begin{aligned} |\nabla_{i_{k+1}} F_k| &\leqslant (E_{i_k} E_{i_{k-1}} |\nabla_{i_{k+1}} F_{k-2}|^2)^{1/2} \\ &\quad + C_{i_{k+1} i_{k-1}} (E_{i_k} E_{i_{k-1}} |\nabla_{i_{k-1}} F_{k-2}|^2)^{1/2} \\ &\quad + C_{i_{k+1} i_k} (E_{i_k} |\nabla_{i_k} F_{k-1}|^2)^{1/2}. \end{aligned} \tag{1.21}$$

Let us observe that such a procedure of expanding always the term containing $\nabla_{i_{k+1}}$ terminates after the $(k-l)$th step. This is because in that step we have

$$\nabla_{i_{k+1}} F_l \equiv \nabla_{i_{k+1}} (E_{i_l} \cdots E_{i_1} f^2)^{1/2} = 0 \tag{1.22}$$

for $i_{k+1} = i_l$ (which is a consequence of the definition of local specification). Hence we get

$$|\nabla_{i_{k+1}} F_k| \leqslant \sum_{l < m \leqslant k} C_{i_{k+1} i_m} (E_{i_k} \cdots E_{i_1} |\nabla_{i_m} F_{m-1}|^2)^{1/2}. \tag{1.23}$$

This procedure shows also that for a fixed $i, j \in \mathbf{\Gamma}$ and $k, k' \in \mathbf{Z}^+$ such that

$$i_{k+1} = j = i_{k'+1} \tag{1.24}$$

the sets of paths defining $\lambda_{ji}^{(k+1)}$ resp. $\lambda_{ji}^{(k'+1)}$ are different. Using this we can termwise identify

$$\sum_{k \geqslant 0} (\lambda_{i_{k+1} i}^{(k+1)}) \bigg|_{i_{k+1} = j} = \chi_{ji} \tag{1.25}$$

with χ being the Green function of the random walk on $\mathbf{\Gamma}$ with transition probabilities C_{ji}, i.e.,

$$\chi_{ji} = \sum_{n \geqslant 0} (C^n)_{ji}. \tag{1.26}$$

(This is similar to Dobrushin theory [7–11].) From (0.13) we have also that

$$0 < \sum_{j \in \mathbf{\Gamma}} \chi_{ji} \leqslant (1-\gamma)^{-1}. \tag{1.27}$$

Therefore, using (1.20), (1.24)–(1.27) we obtain

$$\sum_{k \in \mathbf{N}} \mu\, |\nabla_{i_{k+1}} (E_{i_k} \cdots E_{i_1} f^2)^{1/2}|^2 \leqslant (1-\gamma)^{-2} \mu\, |\nabla f|^2 \tag{1.28}$$

which ends the proof of Lemma 1.1. ∎

To finish the proof of the log-Sobolev inequality (1.1) it is now enough to show that (1.6) is true. Here we give a proof of Lemma 1.2 under the additional assumption that M is a compact Riemmanian manifold. The more general case is considered in Section 2.

Proof of Lemma 1.2. Let f be a (strictly positive) element of $\mathscr{C}_b^1(\Omega)$ and suppose that $f \in \Sigma_{\tilde{\Lambda}}$ for some $\tilde{\Lambda} \in \mathscr{F}_0$. For such a function we define the following norm of its gradient

$$\|\nabla f\| \equiv \sum_{i \in \mathbf{\Gamma}} \|\, |\nabla_i f| \,\|_\infty. \tag{1.29}$$

We extend this definition to all functions $f \in \mathscr{C}_b^1(\Omega)$ for which the sum on the rhs of (1.29) is finite. When M is a compact Riemmanian manifold to prove Lemma 1.2 it is sufficient to show that

$$\lim_{k \to \infty} \|\nabla F_k\| \equiv \lim_{k \to \infty} \|\nabla(E_{i_k} \cdots E_{i_1} f^2)^{1/2}\| = 0. \tag{1.30}$$

This is because we have for any $l > k$

$$\begin{aligned} &\sup_{\omega \in \Omega} |F_l^2(\omega) - F_k^2(\omega)| \\ &\quad = \sup_{\omega \in \Omega} |E_{i_l}^{\omega} \cdots E_{i_{k+1}}^{\cdot}(F_k^2(\cdot) - F_k^2(\omega))| \\ &\quad \leqslant \sup_{\omega, \tilde{\omega} \in \Omega} |F_k^2(\omega) - F_k^2(\tilde{\omega})| \leqslant 2\, \|F_k\|_\infty \sup_{\omega, \tilde{\omega} \in \Omega} |F_k(\omega) - F_k(\tilde{\omega})| \end{aligned} \tag{1.31}$$

and

$$\sup_{\omega, \tilde{\omega} \in \Omega} |F_k(\omega) - F_k(\tilde{\omega})| \leqslant \sum_{i \in \Gamma} \sup_{\substack{\omega, \tilde{\omega} \in \Omega \\ \omega_j = \tilde{\omega}_j,\, j \neq i}} |F_k(\omega) - F_k(\tilde{\omega})| \leqslant C\, \|\nabla F_k\| \tag{1.32}$$

with some constant $0 < C < \infty$ independent of $k \in \mathbf{N}$. Therefore (1.30) shows that the sequence $\{F_k^2 \equiv E_{i_k} \cdots E_{i_1} f^2\}_{k \in \mathbf{N}}$ converges. Additionally since μ is a Gibbs measure we have

$$\mu f^2 - F_k^2(\omega) = \mu(F_k^2(\cdot) - F_k^2(\omega)) \tag{1.33}$$

and using (1.32) together with (1.30) we get (1.6). Let us note that these arguments show also the uniqueness of the measure μ (for the case of compact manifold).

Let us now prove (1.30). To do that we first take a set $\Lambda \in \mathscr{F}_0$ containing $\tilde{\Lambda}$ and show that the sum

$$\sum_{i \in \Lambda} \|\, |\nabla_i F_k|\, \|_\infty \tag{1.34}$$

can be done arbitrarily small by taking $k \in \mathbf{N}$ sufficiently large. Given $n \in \mathbf{N}$, let $k \in \mathbf{N}$ be a number such that each point $i \in \Lambda$ is contained in a finite sequence $\{i_1, ..., i_k\}$ at least n-times. Let $n(i)$ be a biggest number such that $i_{n(i)} = i$. Then using the similar arguments as in (1.14) and (1.21)–(1.23) based on the application of condition (Cii) and definition of local specification we obtain

$$|\nabla_i F_k| \leqslant \sum_{n(i) < m \leqslant k} C_{i i_m} (E_{i_k} \cdots E_{i_m} |\nabla_{i_m} F_{m-1}|^2)^{1/2}. \tag{1.35}$$

From this and (1.15) we get

$$|\nabla_i F_k| \leqslant \sum_{n(i) < m \leqslant k} C_{ii_m} \sum_{j \in \Gamma} \lambda_{i_m j}^{(m)} (E_{i_k} \cdots E_{i_1} |\nabla_j f|^2)^{1/2}$$

$$\leqslant \sum_{n(i) < m \leqslant k} C_{ii_m} \sup_{j \in \tilde{\Lambda}} \lambda_{i_m j}^{(m)} \|\nabla f\|. \tag{1.36}$$

Now given $\varepsilon > 0$ let us choose a set $\Lambda_1 \in \mathscr{F}_0$, $\Lambda \subset \Lambda_1$ such that

$$\sup_{i \in \Lambda} \sum_{i_m \in \Lambda_1^c} C_{ii_m} < \varepsilon. \tag{1.37}$$

Using this, (0.13), and (1.19) we see that

$$\sum_{i \in \Lambda} \sum_{n(i) < m \leqslant k : i_m \in \Lambda_1^c} C_{ii_m} \sup_{j \in \tilde{\Lambda}} \lambda_{i_m j}^{(m)} \leqslant \varepsilon\, |\Lambda|\, (1 - \gamma)^{-1}. \tag{1.38}$$

On the other hand using (0.13) and our assumption that $n(i) \geqslant n$ for $i \in \Lambda$ we get

$$\sum_{i \in \Lambda} \sum_{n(i) < m \leqslant k : i_m \in \Lambda_1} C_{ii_m} \sup_{j \in \tilde{\Lambda}} \lambda_{i_m j}^{(m)} \leqslant |\Lambda|\, \gamma \sup_{m > n} \sup_{\substack{j \in \tilde{\Lambda} \\ i \in \Lambda_1}} \lambda_{ij}^{(m)}. \tag{1.39}$$

By taking $n \in \mathbf{N}$ sufficiently large we can always satisfy the inequality

$$\sup_{m > n} \sup_{\substack{j \in \tilde{\Lambda} \\ i \in \Lambda_1}} \lambda_{ij}^{(m)} < \varepsilon \gamma^{-1} (1 - \gamma)^{-1}. \tag{1.40}$$

Combining (1.39), (1.40) together with (1.36) we obtain

$$\sum_{i \in \Lambda} |\nabla_i F_k| \leqslant \varepsilon 2\, |\Lambda|\, (1 - \gamma)^{-1} \|\nabla f\|. \tag{1.41}$$

Let us now consider the other part of $\|\nabla F_k\|$ connected to Λ^c. Using (Cii) and the fact that $\tilde{\Lambda} \subset \Lambda$ we get

$$\sum_{i \in \Lambda^c} |\nabla_i F_k| \leqslant \sum_{i \in \Lambda^c} \sum_{i' \in \Gamma} C_{ii'} \sup_{j \in \tilde{\Lambda}} \gamma_{i'j} \|\nabla f\|. \tag{1.42}$$

It follows from (0.13) and (1.27) that the rhs of (1.42) can be done arbitrarily small by choosing $\Lambda \in \mathscr{F}_0$, $\tilde{\Lambda} \subset \Lambda$ sufficiently large. This together with (1.41) shows (1.30) and ends the proof of Lemma 1.2. ∎

2. Existence and Uniqueness of Gibbs Measure for Noncompact M

Let $\mathscr{E} = \{E_\Lambda\}_{\Lambda \in \mathscr{F}}$ be a differentiable local specification satisfying condition (Cii). In this section we assume the single spin space M to be a noncompact Riemannian manifold and we would like to consider the question: When does the corresponding set of Gibbs measures $\mathscr{G}(\mathscr{E})$ contain exactly one element? If M is noncompact this question is especially nontrivial. Then we know (see, e.g., [20]), that even in Dobrushin theory a condition corresponding to our (Cii) and (0.13) is not sufficient for $\sharp\mathscr{G}(\mathscr{E})=1$. One has to complete it by adding a restriction on a set of considered measures by imposing a suitable growth condition on the moments. In our situation we shall do the same and we propose to do that as follows.

Let $s(\omega_i, \omega_i')$ denote the Riemannian distance between the points $\omega_i, \omega_i' \in M$. We assume that there is $\bar{\omega} \in \Omega$ such that for some $0<a<\infty$ and any $i \in \Gamma$

$$E_i^{\bar{\omega}} s^2(\bar{\omega}_i, \cdot) \leqslant a^2. \tag{2.1}$$

For this fixed $\bar{\omega} \in \Omega$ and $i \in \Gamma$ we introduce a Σ_i-measurable function

$$s_i(\omega) \equiv s(\bar{\omega}_i, \omega_i) \tag{2.2}$$

which is well defined and differentiable everywhere on M but at most a submanifold of dimension $\dim M - 1$. It is also convenient to impose the following technical condition for any $i \in \Gamma$

$$|\partial s_i| < b, \qquad E_i^{\omega} - \text{a.e.} \tag{2.3a}$$

satisfied for all $\omega \in \Omega$ with some $0<b<\infty$ such that

$$0 < \gamma b < 1. \tag{2.3b}$$

Let us note that the restriction (2.3) is satisfied in a large class of local specifications which are interesting, e.g., for field theory. Using this condition we will show the following result.

THEOREM 2.1. *Suppose $\mathscr{E} = \{E_\Lambda^{\cdot}\}_{\Lambda \in \mathscr{F}}$ is a differentiable local specification satisfying condition* (Cii) *and* (0.13) *supplemented by requirements* (2.1) *and* (2.3) *if the single spin space M is noncompact. Then there is only one Gibbs measure $\mu \in \mathscr{G}(\mathscr{E})$ satisfying the bound*

$$\mu s_i \leqslant A \tag{2.4}$$

with a constant $0 < A < \infty$ independent of $i \in \Gamma$.

Proof. First of all we have to show for noncompact M, that a Gibbs measure μ satisfying (2.4) exists. For that we need the following lemma.

LEMMA 2.2. *Suppose a local differentiable specification* $\mathscr{E} = \{E^{\cdot}_{\Lambda}\}_{\Lambda \in \mathscr{F}}$ *satisfies condition* (Cii) *together with* (2.1) *and* (2.3). *Then for any* $i \in \Gamma$ *and* $k \in \mathbf{N}$

$$E_i^{\omega} s_i \leqslant a + \sum_{j \in \Gamma \setminus i} s_j(\omega)\, C_{ji} b. \tag{2.5}$$

Proof of Lemma 2.2. Let us note that from the Schwarz inequality we have

$$E_i s_i \leqslant (E_i s_i^2)^{1/2}. \tag{2.6}$$

By the fundamental theorem of calculus on manifold with $j \in \Gamma$, $j \neq i$, we get

$$(E_i s_i^2)^{1/2}(\omega) = (E_i s_i^2)^{1/2}(\omega)|_{\omega_j = \bar{\omega}_j} + \left(\int_{\bar{\omega} \to \omega} dx_j \cdot \nabla_j (E_i s_i^2)^{1/2}\right)(\omega), \tag{2.7}$$

where $\bar{\omega}_j \to \omega_j$ indicates integration over a shortest path connecting $\bar{\omega}_j$ and ω_j. The rhs of (2.7) can be bounded with use of (Cii) as

$$(E_i s_i^2)^{1/2}(\omega) \leqslant (E_i s_i^2)^{1/2}(\omega)|_{\omega_j = \bar{\omega}_j} + \left(\int_{\bar{\omega} \to \omega} ds_j\, |\nabla_j (E_i s_i^2)^{1/2}|\right)(\omega)$$

$$\leqslant (E_i s_i^2)^{1/2}(\omega)|_{\omega_j = \bar{\omega}_j} + \left(\int_{\bar{\omega} \to \omega} ds_j C_{ji} (E_i\, |\nabla_i s_i|^2)^{1/2}\right)(\omega). \tag{2.8}$$

Hence, using our technical assumption (2.3a), we obtain

$$(E_i s_i^2)^{1/2}(\omega) \leqslant (E_i s_i^2)^{1/2}(\omega)|_{w_j = \bar{\omega}_j} + s_j(\omega)\, C_{ji} b. \tag{2.9}$$

Now we apply the same consideration to the first term on the rhs of (2.9), with others $j' \neq i, j$. An iteration of this argument and use of assumption (2.1) lead to the inequality

$$(E_i s_i^2)^{1/2}(\omega) \leqslant (E_i s_i^2)^{1/2}(\bar{\omega}) + \sum_{j \in \Gamma \setminus i} s_j(\omega)\, C_{ji} b \leqslant a + \sum_{j \in \Gamma \setminus i} s_j(\omega)\, C_{ji} b. \tag{2.10}$$

This ends the proof of Lemma 2.2. ∎

Now we take a measure $E_{\Lambda}^{\bar{\omega}} \in \mathscr{E}$ and apply the compatibility condition for specifications together with Lemma 2.2 to get for any $i \in \Lambda$

$$E_{\Lambda}^{\bar{\omega}} s_i \leqslant a + \sum_{j \in \Lambda \setminus i} E_{\Lambda}^{\bar{\omega}} s_j C_{ji} b. \tag{2.11}$$

We used here also the fact that by definition of local specification and of function s_i in (2.2) we have for $j \in \Lambda^c$

$$E_{\Lambda}^{\bar{\omega}} s_j = s_j(\bar{\omega}) = 0.$$

From (2.11) and assumption (2.3b) we obtain

$$E_{\Lambda}^{\bar{\omega}} s_i \leqslant a(1-\gamma b)^{-1} \tag{2.12}$$

for any $i \in \Lambda$. Since the functions s_i, $i \in \Gamma$, are bicompact, we may conclude that a sequence of measures $\{E_{\Lambda}^{\bar{\omega}}\}_{\Lambda \in \mathscr{F}_0}$, with $\mathscr{F}_0$ being a countable base in Γ, is compact. Therefore there is a subsequence $\mathscr{F}'_0 \subset \mathscr{F}_0$ such that the limit

$$\lim_{\mathscr{F}'_0} E_{\Lambda}^{\bar{\omega}} \equiv \bar{\mu} \tag{2.13}$$

exists. By construction $\bar{\mu} \in \mathscr{G}(\mathscr{E})$ and satisfies condition (2.4) with

$$A = a(1-\gamma b)^{-1}. \tag{2.14}$$

We will show that the Gibbs measure $\bar{\mu}$ is unique and satisfies

$$\bar{\mu} f = \lim_{k \to \infty} (E_{i_k} \cdots E_{i_1} f)(\bar{\omega}) \tag{2.15}$$

with the sequence $\mathbf{I} \equiv \{i_k \in \Gamma\}_{k \in \mathbf{N}}$ defined in Section 1. Suppose $\mu \in \mathscr{G}(\mathscr{E})$ is a measure satisfying (2.4). To get the uniqueness result we use the definition of Gibbs measure and show that for any $\tilde{\Lambda} \in \mathscr{F}$ and any bounded positive function $f \in \mathscr{C}_b^1(\Omega)$, $f \in \Sigma_{\tilde{\Lambda}}$, we have

$$\lim_{k \to \infty} \mu(E_{i_k} \cdots E_{i_1} f^2(\omega) - E_{i_k} \cdots E_{i_1} f^2(\bar{\omega})) = 0. \tag{2.16}$$

To prove (2.16) it is sufficient to show that in $L_1(\mu)$ we have

$$\lim_{k \to \infty} |(E_{i_k} \cdots E_{i_1} f^2)^{1/2}(\omega) - (E_{i_k} \cdots E_{i_1} f^2)^{1/2}(\bar{\omega})| = 0. \tag{2.17}$$

To consider the integrand in (2.17), for $j \in \Gamma$ and $\omega \in \Omega$, let us define $\omega^{(j)} \in \Omega$ by

$$\omega_l^{(j)} := \begin{cases} \bar{\omega}_l & \text{for } l \prec j \\ \omega_l & \text{for } l \succcurlyeq j. \end{cases} \tag{2.18}$$

With this notation, by application of the fundamental theorem of calculus, we obtain

$$(E_{i_k} \cdots E_{i_1} f^2)^{1/2}(\omega) = (E_{i_k} \cdots E_{i_1} f^2)^{1/2}(\bar{\omega}) + \sum_{j \in \Gamma} \int_{\bar{\omega} \to \omega} dx_j \nabla_j (E_{i_k} \cdots E_{i_1} f^2)^{1/2}(\omega^{(j)}). \tag{2.19}$$

Hence

$$|(E_{i_k} \cdots E_{i_1} f^2)^{1/2}(\omega) - (E_{i_k} \cdots E_{i_1} f^2)^{1/2}(\bar{\omega})| \leqslant \sum_{j \in \Gamma} s_j(\omega) \, \| \, |\nabla_j (E_{i_k} \cdots E_{i_1} f^2)^{1/2}| \, \|_\infty \equiv \sum_{j \in \Gamma} s_j(\omega) \, \| \, |\nabla_j F_k| \, \|_\infty, \tag{2.20}$$

where on the rhs we used (1.12). From this using (2.4), the Hölder inequality, and definition (1.29) we get

$$\begin{aligned}\mu\, &|(E_{i_k}\cdots E_{i_1} f^2)^{1/2}(\omega) - (E_{i_k}\cdots E_{i_1} f^2)^{1/2}(\bar{\omega})| \\ &\leqslant A \sum_{j\in\Gamma} \|\, |\nabla_j F_k|\, \|_\infty \equiv A\, \|\nabla F_k\|. \end{aligned} \tag{2.21}$$

This together with (1.30) shows the uniqueness of the Gibbs measure μ in the class of Gibbs measures satisfying condition (2.4).

Remark. Using condition (Cii) and ideas given above one can obtain estimates on decay of correlations similarly as in Dobrushin theory (see [10, 11, 9]).

3. Applications

In this section we would like to show that our condition (C) is satisfied for a large class of local differentiable specifications $\mathscr{E} = \{E_\Lambda\}_{\Lambda\in\mathscr{F}}$ on (Ω, Σ). A typical construction of such specification goes as follows: We take a continuous probability measure ρ on a finite dimensional (smooth connected) Riemannian manifold M, which satisfies the log-Sobolev inequality with a constant $0 < \bar{c} < \infty$. Such measures are relatively well characterized in the literature (see the references in [50]). We will assume that the measure ρ does not vanish on open sets in M. To avoid some additional discussion we will assume also that M has empty boundary and that the Dirichlet form associated to ρ and defined for smooth compactly supported functions on M is closable (for related conditions see, e.g., [21] and references therein). Using the measure ρ we define a free measure μ_0 on (Ω, Σ) by

$$\mu_0 \equiv \rho^{\otimes\Gamma}. \tag{3.1}$$

By the very definition the free measure μ_0 also satisfies log-S with the same coefficient $\bar{c}$ (see [1]). The conditional expectation with respect to the σ-algebra Σ_{Λ^c}, $\Lambda\in\mathscr{F}$, associated to the free measure will be denoted by $\mu_0|_\Lambda$. Let $\mathscr{A}$ denote the set of real measurable functions on (Ω, Σ).

An interaction Φ is a map

$$\Phi: \mathscr{F} \to \mathscr{A} \tag{3.2}$$

such that for any $X\in\mathscr{F}$

$$\Phi_X \in \Sigma_X. \tag{3.3}$$

We will say that an interaction Φ is *Gibbsian* iff the norm

$$\|\Phi\| \equiv \sup_{i \in \Gamma} \sum_{\substack{X \in \mathscr{F} \\ i \in X}} \|\Phi_X\|_\infty \tag{3.4}$$

is finite. An interaction Φ is called differentiable if $\Phi_X \in \mathscr{C}^1(\Omega)$ for any $X \in \mathscr{F}$. The interaction energy U_Λ at a volume $\Lambda \in \mathscr{F}$ is defined by

$$U_\Lambda \equiv \sum_{X \in \mathscr{F} : X \cap \Lambda \neq \varnothing} \Phi_X. \tag{3.5}$$

Suppose for any $\Lambda \in \mathscr{F}$ and $\omega \in \Omega$

$$0 < \delta_\omega \mu_0|_\Lambda \, e^{-U_\Lambda} < \infty, \tag{3.6}$$

where δ_ω is the point measure concentrated at ω. Then we define a local specification $\mathscr{E} = \{E^{\cdot}_\Lambda\}_{\Lambda \in \mathscr{F}}$ by setting

$$E^\omega_\Lambda(f) := \delta_\omega \frac{\mu_0|_\Lambda (e^{-U_\Lambda} f)}{\mu_0|_\Lambda (e^{-U_\Lambda})}. \tag{3.7}$$

For $\Lambda \equiv \{i\}$, with $i \in \Gamma$, we will write simply E_i instead of $E_{\{i\}}$.

A first interesting (especially for applications in statistical mechanics) situation is described in the following proposition.

PROPOSITION 3.1. *Let Φ be a differentiable Gibbsian interaction such that for any $i, j \in \Gamma$*

$$\|\nabla_j U_i\|_\infty < \infty. \tag{3.8}$$

Then the corresponding local differentiable specification $\mathscr{E} = \{E^{\cdot}_\Lambda\}_{\Lambda \in \mathscr{F}}$ defined by (3.7) *satisfies condition* (C) *with a uniform* log-Sobolev *constant $0 < c_0 < \infty$ satisfying*

$$0 < c_0 \leqslant \bar{c} e^{2\|\Phi\|} \tag{3.9}$$

and

$$0 \leqslant C_{ji} \leqslant 2^{-1/2} c_0^{1/2} \sup_{\omega, \tilde{\omega} \in \Omega} |\nabla_j U_i(\omega) - \nabla_j U_i(\tilde{\omega})| \tag{3.10}$$

for $i, j \in \Gamma, i \neq j$ (for $i = j$ we have $C_{ij} \equiv 0$).

Proof. The differentiability of local specification $\mathscr{E}$ corresponding to interaction Φ and defined by (3.7) easily follows from differentiability of Φ and assumption (3.8). The inequality (3.9), i.e., the condition (Ci), is a simple consequence of the Holley–Stroock lemma (Lemma 5.1 in [17]).

Using the condition (Ci), we show in Lemma 3.2 below that also condition (Cii) is satisfied with a corresponding matrix $\mathbf{c} = \{C_{ij}\}_{i,j \in \Gamma}$ fulfilling (3.10) for off diagonal elements. This ends the proof of Proposition 3.1. ∎

LEMMA 3.2. *Assume that*

$$\|\nabla_j U_i\|_\infty < \infty \tag{3.11}$$

and that the measures E_i^ω *given by* (3.7) *fulfill* log-S *with coefficient* $0 < c_0 < \infty$ *independent of* $i \in \Gamma$ *and* $\omega \in \Omega$.

Then for any strictly positive function $f \in \mathscr{C}_b^1(\Omega)$ *and any* $i, j \in \Gamma$, $i \neq j$, *the inequality*

$$|\nabla_j (E_i^\omega f^2)^{1/2}| \leqslant (E_i^\omega |\nabla_j f|^2)^{1/2} + C_{ji} (E_i^\omega |\nabla_i f|^2)^{1/2} \tag{3.12}$$

is satisfied with

$$0 \leqslant C_{ji} \leqslant 2^{-1/2} c_0^{1/2} \sup_{\omega, \tilde{\omega} \in \Omega} |\nabla_j U_i(\omega) - \nabla_j U_i(\tilde{\omega})|. \tag{3.13}$$

Proof. Let us consider a strictly positive function $f \in \mathscr{C}_b^1(\Omega)$. From the definition of our local specification we have

$$E_i^\omega f^2 > 0 \tag{3.14}$$

and

$$(E_i f^2)^{1/2} \in \mathscr{C}_b^1(\Omega). \tag{3.15}$$

Therefore

$$|\nabla_j (E_i f^2)^{1/2}| = 2 (E_i f^2)^{-1/2} |\nabla_j E_i f^2|. \tag{3.16}$$

On the other hand

$$\nabla_j E_i f^2 = E_i 2f \nabla_j f + E_i (f^2, \nabla_j U_i), \tag{3.17}$$

where we used the notation

$$E_i(F, G) \equiv E_i(FG) - E_i(F)\, E_i(G). \tag{3.18}$$

From (3.17) we have

$$|\nabla_j E_i f^2| \leqslant |E_i 2f \nabla_j f| + |E_i (f^2, \nabla_j U_i)|. \tag{3.19}$$

Using Hölder inequality we get

$$|E_i 2f \nabla_j f| \leqslant 2 (E_i f^2)^{1/2} (E_i |\nabla_j f|^2)^{1/2}. \tag{3.20}$$

This by use of (3.16) yields the first term from the rhs of (3.12). To estimate the second term from the rhs of (3.19) we use the identity

$$E_i(f^2, \nabla_j U_i) \equiv \tfrac{1}{2} E_i \otimes \tilde{E}_i (f^2(\omega) - f^2(\tilde{\omega}))(\nabla_j U_i(\omega) - \nabla_j U_i(\tilde{\omega})) \tag{3.21}$$

with ω resp. $\tilde{\omega}$ being the integration variables with respect to E_i resp. its isomorphic copy $\tilde{E}_i$. From this we obtain

$$|E_i(f^2, \nabla_j U_i)| \leqslant \tfrac{1}{2} \sup_{\omega, \tilde{\omega} \in \Omega} |\nabla_j U_i - \nabla_j \tilde{U}_i| \cdot E_i \otimes \tilde{E}_i \, |f^2 - \tilde{f}^2|, \tag{3.22}$$

where we shorten the notation by writing $f \equiv f(\omega)$ and $\tilde{f} \equiv f(\tilde{\omega})$, and similarly with U_i. Now by Hölder and the triangle inequality we obtain

$$E_i \otimes \tilde{E}_i \, |f^2 - \tilde{f}^2| \leqslant 2(E_i f^2)^{1/2} \, (E_i \otimes \tilde{E}_i (f - \tilde{f})^2)^{1/2}. \tag{3.23}$$

Since E_i satisfies log-S with a constant $0 < c_0 < \infty$, so $E_i \otimes \tilde{E}_i$ also does with the same constant. Then, because obviously

$$E_i \otimes \tilde{E}_i (f - \tilde{f}) = 0 \tag{3.24}$$

we may use the mass gap inequality [23–25]

$$E_i \otimes \tilde{E}_i (f - \tilde{f})^2 \leqslant c_0 E_i \otimes \tilde{E}_i (|\nabla_i f|^2 + |\nabla_i \tilde{f}|^2) = 2c_0 E_i \, |\nabla_i f|^2 \tag{3.25}$$

together with (3.23) to obtain

$$E_i \otimes E_i \, |f^2 - \tilde{f}^2| \leqslant 2(E_i f^2)^{1/2} \, 2^{1/2} c_0^{1/2} (E_i \, |\nabla_i f|^2)^{1/2}. \tag{3.26}$$

Inserting this into (3.22) we get

$$|E_i(f^2, \nabla_j U_i)| \leqslant 2(E_i f^2)^{1/2} \, [2^{-1/2} c_0^{1/2} \sup_{\omega, \tilde{\omega}} |\nabla_j U_i - \nabla_j \tilde{U}_i|] \cdot (E_i \, |\nabla_i f|^2)^{1/2}. \tag{3.27}$$

That by use of (3.16) yields the second term on the rhs of (3.12) with C_{ji} satisfying (3.13). This ends the proof of Lemma 3.2. ∎

We would like to mention the following explicit example important for applications in statistical mechanics.

EXAMPLE 3.3. Let $(M = S^N, g)$ be an $N \in \mathbf{N}$ dimensional sphere with a C^1 metric g. In this case the uniform probability measure ρ on M satisfies the log-Sobolev inequality; for $N = 1$ see [26, 23, 16, 27], for $N \geqslant 2$ see [28, 2]. Then assuming a smooth interaction to be sufficiently small, one may show that the corresponding measures E_i^ω satisfy the log-Sobolev inequality with a constant $0 < c_0 < \infty$ (decreasing with interaction) independent of $i \in \Gamma$ and $\omega \in \Omega$; for $N \in \mathbf{N}$ by arguments given in the proof of

Proposition 3.1 and used in [29] for $N=1$, and for $N \geqslant 2$ by Γ_2-criterion, see in [3]. This by Lemma 3.2 implies condition (Cii) and we may apply Theorem 0.1 to obtain the log-Sobolev inequality for corresponding (unique) Gibbs measure on infinite lattice $\mathbf{\Gamma}$. Let us note that log-Sobolev inequalities for infinite systems corresponding to $M=S^N$, $N \geqslant 2$, have been obtained by use of the Bakry–Emery criterion in [3]. A particular example of application of our method with $N=1$ and nearest neighbors interaction on $\mathbf{\Gamma} \equiv \mathbf{Z}^d$ has been presented in [29], where the restriction to nearest neighbor interactions allowed a compact presentation.

Recently the case $M=S^1$ has been solved also by the Γ_2-criterion in [4].

Let us consider the second situation, when the single spin space is non-compact and when the interaction can be unbounded with unbounded derivative in general. Such a situation appears in field theory and also in statistical mechanics, where we need to consider a single spin space $M \equiv \mathbf{R}$ and where we are interested in the study of infinite volume measures defined as a local perturbation of a Gaussian measure. To put this situation into our setting let us introduce a positive definite real matrix $G \equiv \{G_{ij}\}_{i,j \in \mathbf{\Gamma}}$ possessing an inverse G^{-1} which satisfies

$$\sum_{j \in \mathbf{\Gamma}} |G_{ij}^{-1}| < \infty. \tag{3.28}$$

For simplicity let us assume that for any $i \in \mathbf{\Gamma}$

$$G_{ii}^{-1} = G_{00}^{-1}. \tag{3.29}$$

Then we define a measure

$$\rho(\cdot) := \left(\frac{G_{00}^{-1}}{2\pi}\right)^{1/2} \int dx (e^{-(1/2)\, G_{00}^{-1} x^2} \cdot) \tag{3.30}$$

which satisfies log-S with coefficient $\bar{c} \equiv G_{00}^{-1}$. Let us set

$$\Phi_X \equiv \tfrac{1}{2} G_{ij}^{-1} \omega_i \omega_j \tag{3.31}$$

for $X \equiv \{i, j\}$ and zero for all other sets $X \in \mathscr{F}$, $|X| > 2$.

Additionally we take a local interaction

$$\Phi_{\{i\}} \equiv V_i := V(\omega_i), \qquad i \in \mathbf{\Gamma} \tag{3.32}$$

with a smooth real function V such that

$$\rho(e^{-V}) < \infty. \tag{3.33}$$

In order to define a local specification we shall restrict ourselves to a subset Ω_G of all configurations $\omega \in \Omega$ fulfilling for any $i \in \Gamma$ the restriction

$$\left| \sum_{j \in \Gamma} G_{ij}^{-1} \omega_j \right| < \infty. \tag{3.34}$$

Then we have that

$$U_i(\omega) \equiv \sum_{j \in \Gamma \setminus i} G_{ij}^{-1} \omega_i \omega_j + V(\omega_i) \tag{3.35}$$

is finite for all $\omega \in \Omega_G$ and $i \in \Gamma$. One may now check that the corresponding family $\mathscr{E}_G \equiv \{E_\Lambda^\omega\}_{\Lambda \in \mathscr{F}, \omega \in \Omega_G}$ given (3.7) forms a local differentiable specification. For further purposes let us set

$$\bar{G}_{ji}^{-1} \equiv \frac{G_{ji}^{-1}}{G_{00}^{-1}}. \tag{3.36}$$

The first interesting result in the present case is the following proposition.

PROPOSITION 3.4. *Let $\mathscr{E}_G \equiv \{E_\Lambda^\omega\}_{\Lambda \in \Gamma, \omega \in \Omega_G}$ be a local differentiable specification defined by* (3.7) *and* (3.30)–(3.35). *Suppose there is $0 \leqslant m^2 < \infty$ such that in the sense of quadratic forms*

$$0 < G^{-1} - m^2 < \infty \tag{3.37}$$

and the function $(1/2)\, m^2 x^2 + V(x)$ is convex. Then the corresponding probability measures E_i^ω satisfy log-S *with a constant $0 < c_0 < (G_{00}^{-1} - m^2)$ independent of $i \in \Gamma$ and $\omega \in \Omega_G$. Additionally if*

$$\|\partial V\|_\infty < \infty \tag{3.38}$$

then the specification $\mathscr{E}_G$ fulfils condition (Cii) *with*

$$0 \leqslant C_{ji} \leqslant |\bar{G}_{ji}^{-1}| \, (1 + 2^{-1/2} c_0^{1/2} \sup_{x, y \in \mathbf{R}} |V'(x) - V'(y)|) \tag{3.39}$$

for $i, j \in \Gamma$, $i \neq j$.

Remark. One may see that the situation described in the above Proposition 3.4 corresponds to Dobrushin uniqueness, see, e.g., [30].

Proof. The proof of condition (Ci) is easy, e.g., by use of the Bakry–Emery criterion [2] or any other standard argumens for finite dimensional measures.

To see that (Cii) holds we use the same strategy as in the proof of Lemma 3.2. Therefore we take a positive function $f \in \mathscr{C}_b^1(\Omega)$, points $i, j \in \Gamma$, $i \neq j$, and repeat (3.16)–(3.20). By this we get the first term from the rhs of

(Cii). To analyze the term corresponding to the second one on the rhs of (3.17), we use the present definition of measure E_i and integration by parts with the measure ρ to get

$$E_j(f^2, \nabla_j U_i) = E_i(f^2, G_{ji}^{-1}\omega_i) = \bar{G}_{ji}^{-1} E_i 2f\nabla_i f - \bar{G}_{ji}^{-1} E_i(f^2, \nabla_i V). \tag{3.40}$$

This formula and assumption (3.38) allow us to use the same arguments as in the proof of Lemma 3.2. By this we arrive at the inequality

$$|\nabla_j (E_i f^2)^{1/2}| \leqslant (E_i |\nabla_j f|^2)^{1/2} + C_{ji}(E_i |\nabla_i f|^2)^{1/2} \tag{3.41}$$

with

$$0 \leqslant C_{ji} \leqslant |\bar{G}_{ji}^{-1}| \left(1 + 2^{-1/2} c_0^{1/2} \sup_{x, y \in \mathbf{R}} |V'(x) - V'(y)|\right). \tag{3.42}$$

This ends the proof of Proposition 3.4. ∎

We would like to mention one explicit example of application of the last result, important for statistical mechanics.

EXAMPLE 3.5. Let for $i, j \in \Gamma \equiv \mathbf{Z}^d$

$$G_{ij} := \frac{1}{(2\pi)^d} \int_{(\pi, \pi)^d} d_d q \, e^{iq(i-j)} \hat{G}(q) \tag{3.43}$$

with a symmetric function $\hat{G}(q)$ such that

$$0 < \hat{G}(q) \leqslant \|\hat{G}\|_\infty < \infty. \tag{3.44}$$

Let

$$V(x) = -\ln ch\beta^{1/2} x. \tag{3.45}$$

Then one may show (see [31, 32]) that the local specification and corresponding Gibbs measures describes uniquely a discrete spin Ising model on the lattice. In the situation described by (3.43)–(3.45), with $\hat{G}$ sufficiently smooth if one wishes (3.28) to be true, the assumption (3.38) of Proposition 3.4 is satisfied for

$$\beta \|\hat{G}\|_\infty < 1 \tag{3.46}$$

which is precisely the uniqueness region of [32]. (The last being true also for long range interactions for which (3.28) is not fulfilled.) Then we have the uniform log-S constant

$$c_0 = G_{00}^{-1} - \beta \tag{3.47}$$

and (for $i \neq j$)

$$0 < C_{ji} < |\bar{G}_{ji}^{-1}| \, (1 + (2\beta c_0)^{1/2}). \tag{3.48}$$

This means that our condition (0.13) reads

$$(1 + (2\beta c_0)^{1/2}) \sum_{j \in \Gamma \setminus i} |\bar{G}_{ji}^{-1}| < 1. \tag{3.49}$$

Unfortunately the presented result does not include an important (for applications in euclidean field theory) case when the local interaction V does not satisfy (3.37). In general this case can be very complicated, but for mentioned applications it is sufficient to consider the situation when all off-diagonal elements of the matrix G^{-1} are nonpositive. This last restriction implies, see [6, 41, 42], that the corresponding local specification $\mathscr{E} = \{E^{\cdot}_{\Lambda}\}_{\Lambda \in \mathscr{F}}$ is *attractive*, i.e., for any increasing (with respect to each coordinate ω_i) function $f \in \Sigma$, its image $E^{\cdot}_{\Lambda} f$ is also an increasing function, for any $\Lambda \in \mathscr{F}$. (That property in euclidean field theory corresponds to "second quantization" of maximum principle of classical (quasi-linear) elliptic operators.)

Now suppose that a local interaction V is such that its derivative V' exists and is increasing. Then for any f from a set $\mathscr{A}^{\uparrow}_{+}$, of nonnegative increasing functions on (Ω, Σ), we have that $f^2 \in \mathscr{A}^{\uparrow}_{+}$ and so

$$E_i f^2 \in \mathscr{A}^{\uparrow}_{+} \tag{3.50}$$

for all $E_i \in \mathscr{E}$, $i \in \Gamma$, with $\mathscr{E}$ being our attractive local specification. Using this we get for $i, j \in \Gamma$, $i \neq j$

$$0 \leqslant \nabla_j E_i f^2 = 2E_i f \nabla_j f + |G_{ji}^{-1}| \, E_i(f^2, \omega_i) \tag{3.51}$$

(where we use the definition of E_i and take into account that $G_{ji}^{-1} \leqslant 0$ for $i \neq j$). Note that by FKG inequality, see, e.g., [6, 41, 42], we have

$$0 \leqslant E_i(f^2, \omega_i) \tag{3.52}$$

and

$$0 \leqslant E_i(f^2, V'), \tag{3.53}$$

since by our assumption both f^2 and V' are increasing (nondecreasing). Using these and integration by parts with the measure ρ we obtain

$$\begin{aligned} 0 \leqslant E_i(f^2, \omega_i) &= (G_{00}^{-1})^{-1} \, (E_i 2f \nabla_i f - E_i(f^2, V')) \\ &\leqslant (G_{00}^{-1})^{-1} \, E_i 2f \nabla_i f. \end{aligned} \tag{3.54}$$

From (3.51) and (3.54), by application of the Hölder inequality together with (3.16), we get for any $f \in \mathscr{A}_+^{\uparrow}$ and $i, j \in \Gamma$, $i \neq j$ the inequality

$$0 \leqslant \nabla_j (E_i f^2)^{1/2} \leqslant (E_i |\nabla_j f|^2)^{1/2} + |\bar{G}_{ji}^{-1}| (E_i |\nabla_i f|^2)^{1/2} \tag{3.55}$$

with $\bar{G}_{ji}^{-1}$ defined in (3.36). This inequality and the fact that

$$E_i(\mathscr{A}_+^{\uparrow}) \subseteq \mathscr{A}_+^{\uparrow} \tag{3.56}$$

for any $i \in \Gamma$, allow us to apply the arguments of Section 2 to obtain

PROPOSITION 3.6. *Let $\mathscr{E} = \{E_\Lambda^{\cdot}\}_{\Lambda \in \mathscr{F}}$ be an attractive local specification defined by* (3.7) *and* (3.30)–(3.35), *with a matrix $\{G_{ij}^{-1}\}$ whose off-diagonal elements are nonpositive and a continuous local interaction V, for which V' exists ρ –* a.e. *and is increasing. Suppose that*

$$\gamma = \sup_{i \in \Gamma} \max \left(\sum_{j \in \Gamma \setminus i} |\bar{G}_{ji}^{-1}|, \sum_{j \in \Gamma \setminus i} |\bar{G}_{ij}^{-1}| \right) < 1. \tag{3.57}$$

Then the unique Gibbs measure $\mu \in \mathscr{G}(\mathscr{E})$ with the bounded moments satisfies for any function $f \in \mathscr{A}_+^{\uparrow}$ a log-*Sobolev inequality*

$$\mu f^2 \log |f| \leqslant c\mu |\nabla f|^2 + \mu f^2 \log(\mu f^2)^{1/2} \tag{3.58}$$

with a constant $0 < c < \infty$ independent of $f \in \mathscr{H}_+(\mu)$.

Remark. We expect that Proposition 3.6 in fact implies log-S for the unique measure $\mu \in \mathscr{G}(\mathscr{E})$ for an attractive specification $\mathscr{E}$.

As an example of an application of Proposition 3.6, let us mention the lattice approximation of exponential model [34] of euclidean field theory in two dimensions. In this case we have

$$V(x) = \lambda \int dv(a)\, e^{ax} \tag{3.59}$$

with v a probability measure on $\mathbf{R}$ and $0 < \lambda < \infty$. Using our result and taking the continuum limit [34, 41, 42], of lattice model (with defining objects suitably dependent on lattice spacing) we obtain the (restricted) log-Sobolev inequality for the unique [44] Gibbs measure of exponential model in two dimensional euclidean space with corresponding constant $0 < c \leqslant m_0^{-2}$, with m_0 being the bare mass of the model.

Let us note that the result corresponding to the last remark and Proposition 3.6 may also be obtained by use of the Bakry–Emery criterion [2] with $V \in \mathscr{C}^2(M)$ convex (without restriction to the functions $f \in \mathscr{A}_+^{\uparrow}$).

By this we finish the study of the continuous case, when the single spin space M is a smooth Riemannian manifold.

4. Log-Sobolev Inequalities for Discrete Spin Systems

In this section the single spin space is taken to be $M=\{-1,+1\}$, which is a case of great importance for statistical mechanics. The space of spin configurations is now $\Omega=\{-1,+1\}^{\Gamma}$ and following the tradition its elements we will denote by $\sigma\equiv(\sigma_i)_{i\in\Gamma}$. We keep the same notation as before for σ-algebras of sets in Ω. The role of the gradient ∇_i with respect to the ith coordinate is now played by the projection operator

$$B_i f := \tfrac{1}{2}\,(f|_{\sigma_i=+1} - f|_{\sigma_i=-1})\sigma_i. \tag{4.1}$$

Let

$$A_i \equiv 1 - B_i. \tag{4.2}$$

For $\Lambda\subseteq\Gamma$ we define

$$B_\Lambda f\equiv (B_i f)_{i\in\Lambda} \tag{4.3}$$

and

$$|B_\Lambda f|^2 \equiv \sum_{i\in\Lambda} |B_i f|^2. \tag{4.4}$$

If $\Lambda=\Gamma$ we write simply $B\equiv B_\Gamma$.

Let μ be a probability measure on (Ω,Σ). We say that μ satisfies the *log-Sobolev inequality* with a constant $0<c<\infty$ iff

$$\mu f^2 \log|f| \leqslant c\mu\,|Bf|^2 + \mu f^2 \log(\mu f^2)^{1/2} \tag{4.5}$$

for any measurable function f for which $\mu f^2<\infty$ and $\mu\,|Bf|^2<\infty$. Let μ_0 be a free measure defined as the infinite product of uniform probability measures on $M=\{-1,+1\}$. It is known, see [1], that μ_0 satisfies log-S with a constant $c=1$. Suppose $\Phi\equiv\{\Phi_X\}_{X\in\mathscr{F}}$ is a Gibbsian interaction, i.e., an interaction on (Ω,Σ) satisfying the condition

$$\|\Phi\| \equiv \sup_{i\in\Gamma} \sum_{\substack{X\in\mathscr{F}\\ i\in X}} \|\Phi_X\|_\infty < \infty. \tag{4.6}$$

A corresponding local specification $\mathscr{E}=\{E_\Lambda\}_{\Lambda\in\mathscr{F}}$, given similarly as in Section 3 (by (3.5)–(3.7), is well defined.

For investigation of Glauber's stochastic dynamics [12] one uses a semigroup whose generator $\mathscr{L}$ is defined by

$$\mathscr{L}f(\sigma) := \sum_{k\in\Gamma} c_k(\sigma)(f(\sigma^k)-f(\sigma)) \tag{4.7}$$

for any local function, i.e., $f \in \Sigma_\Lambda$ for some $\Lambda \in \mathscr{F}$, with the functions

$$c_k \equiv \frac{1}{2}\left(1 - \frac{B_k e^{-U_k}}{A_k e^{-U_k}}\right), \tag{4.8}$$

where U_k are defined with potential Φ, and

$$(\sigma^k)_i \equiv \begin{cases} \sigma_i & \text{for} \quad i \neq k \\ -\sigma_i & \text{for} \quad i = k. \end{cases} \tag{4.9}$$

One can show, see [13, 14], that

$$\mu \mathscr{L} f = 0 \tag{4.10}$$

and so any Gibbs measure is a stationary measure for the corresponding stochastic dynamics. A quadratic form corresponding to $-\mathscr{L}$ with the Gibbs measure μ equals (see [13])

$$\mu(f(-\mathscr{L}f)) = 2\mu\left(\sum_k c_k \, |B_k f|^2\right). \tag{4.11}$$

Since by our assumption (4.6) and (4.8) we have

$$0 < \tfrac{1}{2}(1 - th \, \|\Phi\|) \leqslant c_k(\sigma) \leqslant 1 \tag{4.12}$$

so the inequality (4.5) is equivalent to the following *log-Sobolev inequality for the Gibbs measure* μ

$$\mu f^2 \log |f| \leqslant c' \mu(f(-\mathscr{L}f)) + \mu f^2 \log(\mu f^2)^{1/2} \tag{4.13}$$

with a constant $0 < c' < \infty$ independent of a function f satisfying $\mu f^2 < \infty$ and $\mu \, |Bf|^2 < \infty$. By general theory of [1] the inequality (4.13) implies hypercontractivity of the semigroup

$$P_t \equiv e^{t\mathscr{L}} \tag{4.14}$$

and a mas gap in the spectrum of generator $-\mathscr{L}$ [25, 23].

In this section we would like to extend our previous results to the case of discrete spins. It appears that in the present situation we shall have to modify the condition (C). We begin by showing how it could look like. Afterwards we prove a result corresponding to Theorem 0.1 in the present situation.

First of all let us note that in the case under consideration the condition (Ci) is essentially redundant.

Using Theorem 3 of [1], which sayss that the value of the log-Sobolev constant of uniform measure on $\{-1, +1\}$ equals one, and the Holley–Stroock Lemma (Lemma 5.1 in [7]) one easily gets

PROPOSITION 4.1. *Let* $\mathscr{E} = \{E^{\cdot}_{\Lambda}\}_{\Lambda \in \mathscr{F}}$ *be a local specification on* (Ω, Σ) *defined with a Gibbsian potential* Φ. *Then the one point measures* E_i^{σ} *satisfy the* log-*Sobolev inequality with a constant* $0 < c_0 < \infty$ *independent of* $i \in \Gamma$ *and* $\sigma \in \Omega$, *satisfying*

$$0 \leqslant c_0 \leqslant e^{2\,\|\Phi\|}. \tag{4.15}$$

Remark. A similar result holds also for other measures $E_{\Lambda}^{\sigma} \in \mathscr{E}$, with the corresponding constants dependent only on the volume $|\Lambda|$ and the norm (4.6) of interaction.

Now we would like to find an analog of condition (Cii). In the present case we have the following result

PROPOSITION 4.2. *Let* $\mathscr{E} = \{E^{\cdot}_{\Lambda}\}_{\Lambda \in \mathscr{F}}$ *be a local specification on* (Ω, Σ) *defined with a Gibbsian potential* Φ. *Then for any function* $f \in \Sigma$ *and any* $i, j \in \Gamma$, $i \neq j$

$$|B_j(E_i f^2)^{1/2}| \leqslant \alpha (E_i\,|B_j f|^2)^{1/2} + C_{ji}(E_i\,|B_i f|^2)^{1/2} \tag{4.16}$$

with

$$1 \leqslant \alpha \leqslant 2^{1/2} e^{12\,\|\Phi\|} \tag{4.17}$$

and

$$0 \leqslant C_{ji} \leqslant 2^{-1/2} e^{8\,\|B_j U_i\|_\infty} c_0^{1/2}\,\|B_j U_i\|_\infty. \tag{4.18}$$

Note that by definition of local specification

$$B_i(E_i f^2)^{1/2} \equiv 0. \tag{4.19}$$

Using this let us introduce a matrix $\mathbf{c} \equiv \{C_{ji}\}_{i,j \in \Gamma}$ with

$$C_{ii} \equiv 0. \tag{4.20}$$

Proof. Let us note that for any function $F \in \Sigma$ we have

$$B_j F^2 = 2A_j F \cdot B_j F. \tag{4.21}$$

From this we see that to get (4.27) it is sufficient to show the bound

$$|B_j(E_i f^2)^{1/2}| \leqslant 2A_j(E_i f^2)^{1/2} \cdot [\text{rhs (4.16)}]. \tag{4.22}$$

By definition (4.1) of B_j we have

$$|B_j E_i f^2| = \tfrac{1}{2}\,|E_i f|^2_{\sigma_j = +1} - E_i f|^2_{\sigma_j = -1}|. \tag{4.23}$$

We would like to study the rhs of (4.23) using the fundamental theorem of calculus. Therefore we introduce interpolating functions

$$f_{s_j}(\sigma) \equiv f(\sigma_{\Gamma \setminus j}, s_j) := A_j f + \bar{B}_j f \cdot s_j, \tag{4.24}$$

where $s_j \in [-1, 1]$ and

$$\bar{B}_j f \equiv \frac{B_j f}{\sigma_j}. \tag{4.25}$$

We will need also interpolating measures E_{i,s_j}, $s_j \in [-1, 1]$ defined by setting in the corresponding definition (3.7) an interpolating interaction energy

$$U_{i,s_j} \equiv U_i(\sigma_{\Gamma\setminus j}, s_j) := A_j U_i + \bar{B}_j U_i \cdot s_j. \tag{4.26}$$

Using this we have

$$|B_j E_i f^2| = \left| \frac{1}{2} \int_{-1}^{1} ds_j\, 2E_{i,s_j}(f_{s_j}\, \partial_j f_{s_j}) + \frac{1}{2} \int_{-1}^{1} ds_j\, E_{i,s_j}(f_{s_j}^2, \partial_j U_{i,s_j}) \right|, \tag{4.27}$$

where $\partial_j \equiv d/ds_j$.

Let us consider the first term from the rhs of (4.27). By the Schwarz inequality we get

$$|E_{i,s_j}(f_{s_j}\, \partial_j f_{s_j})| \leqslant (E_{i,s_j} f_{s_j}^2)^{1/2}\, (E_{i,s_j} |\partial_j f_{s_j}|^2)^{1/2}. \tag{4.28}$$

Since

$$\left| \frac{dE_{i,s_j}}{dE_i} \right| \leqslant e^{4\, \|B_j U_i\|_\infty} \tag{4.29}$$

and by definition (4.24) of the interpolating function

$$\partial_j f_{s_j} = \bar{B}_j f, \tag{4.30}$$

so using (4.28) and (4.25) we obtain

$$|E_{i,s_j}(f_{s_j}\, \partial_j f_{s_j})| \leqslant e^{4\, \|B_j U_i\|_\infty} (E_i f_{s_j}^2)^{1/2}\, (E_i\, |B_j f|^2)^{1/2}. \tag{4.31}$$

Now let us note that by the Hölder inequality we get

$$\frac{1}{2} \int_{-1}^{1} ds_j (E_i f_{s_j}^2)^{1/2} \leqslant (E_i A_j f^2)^{1/2}. \tag{4.32}$$

Using (4.29) we have also

$$\begin{aligned} A_j (E_i f^2)^{1/2} &\equiv \tfrac{1}{2} \left[(E_i|_{\sigma_j = +1} f|_{\sigma_j = +1}^2)^{1/2} + (E_i|_{\sigma_j = -1} f|_{\sigma_j = -1}^2)^{1/2} \right] \\ &\geqslant e^{-2\, \|B_j U_i\|_\infty}\, \tfrac{1}{2} \left[(E_i f_{\sigma_j = +1}^2)^{1/2} + (E_i f|_{\sigma_j = -1}^2)^{1/2} \right] \\ &\geqslant e^{-2\, \|B_j U_i\|_\infty} (E_i\, \tfrac{1}{2} A_j f^2)^{1/2}. \end{aligned} \tag{4.33}$$

From (4.32) and (4.33) we get

$$\frac{1}{2}\int_{-1}^{1} ds_j (E_i f_{s_j}^2)^{1/2} \leqslant 2^{1/2} e^{2\,\|B_j U_i\|_\infty} A_j (E_i f^2)^{1/2}. \tag{4.34}$$

Combining (4.28) and (4.31) we obtain

$$\left|\frac{1}{2}\int_{-1}^{1} ds_j\, 2E_{i,s_j} f_{s_j}\, \partial_j f_{s_j}\right| \leqslant 2A_j (E_i f^2)^{1/2}\, [2^{1/2} e^{6\,\|B_j U_i\|_\infty} (E_i\, |B_j f|^2)^{1/2}]. \tag{4.35}$$

Let us now consider the second term from the rhs of (4.27). We use the property (4.30) for U_{i,s_j} and the identity

$$E_{i,s_j}(f_{s_j}^2, \bar{B}_j U_i) = \tfrac{1}{2} E_{i,s_j} \otimes \tilde{E}_{i,s_j} (f_{s_j}^2 - \tilde{f}_{s_j}^2)(\bar{B}_j U_i - \bar{B}_j \tilde{U}_i), \tag{4.36}$$

where to shorten the notation we introduced for functions a convention $F \equiv F(\sigma)$ and $\tilde{F} \equiv F(\tilde{\sigma})$ with σ resp. $\tilde{\sigma}$ being the integration variables of E_{i,s_j} resp. $\tilde{E}_{i,s_j}$ the isomorphic copy of E_{i,s_j}. From (4.36) and the Hölder inequality we get

$$\begin{aligned}
&|E_{i,s_j}(f_{s_j}^2, \bar{B}_j U_i)| \\
&\quad \leqslant \tfrac{1}{2}\, \|\bar{B}_j U_i - \bar{B}_j \tilde{U}_i\|_\infty \cdot |E_{i,s_j} \otimes \tilde{E}_{i,s_j} (f_{s_j}^2 - \tilde{f}_{s_j}^2)| \\
&\quad \leqslant \|\bar{B}_j U_i - \bar{B}_j \tilde{U}_i\|_\infty\, (E_{i,s_j} f_{s_j}^2)^{1/2}\, (E_{i,s_j} \otimes \tilde{E}_{i,s_j} (f_{s_j} - \tilde{f}_{s_j})^2)^{1/2}.
\end{aligned} \tag{4.37}$$

Now we use (4.29) to increase the rhs of (4.37) as

$$\begin{aligned}
|E_{i,s_j}(f_{s_j}^2, \bar{B}_j U_i)| \leqslant \|\bar{B}_j U_i - \bar{B}_j \tilde{U}_i\|_\infty\, &e^{6\,\|B_j U_i\|_\infty} (E_i f_{s_j}^2)^{1/2} \\
&\cdot (E_i \otimes \tilde{E}_i (f_{s_j} - \tilde{f}_{s_j})^2)^2.
\end{aligned} \tag{4.38}$$

Integration of both sides of (4.38) with respect to s_j and similar arguments as in (4.32) and (4.33)–(4.34), yield the inequality

$$\begin{aligned}
&\left|\frac{1}{2}\int_{-1}^{1} ds_j\, E_{i,s_j}(f_{s_j}^2, \partial_j U_{i,s_j})\right| \\
&\qquad \leqslant 2A_j (E_i f^2)^{1/2}\, 2^{-1/2} e^{8\,\|B_j U_i\|_\infty}\, \|\bar{B}_j U_i - \bar{B}_j \tilde{U}_i\|_\infty \\
&\qquad\quad \times (E_i \otimes \tilde{E}_i\, \tfrac{1}{2} A_j (f - \tilde{f})^2)^{1/2}.
\end{aligned} \tag{4.39}$$

To estimate the last factor on the rhs of (4.39) let us note that

$$\begin{aligned}
&(E_i \otimes \tilde{E}_i\, \tfrac{1}{2} A_j (f - \tilde{f})^2)^{1/2} \\
&\quad = (E_i \otimes \tilde{E}_i [\tfrac{1}{2} (A_j f - A_j \tilde{f})^2 + \tfrac{1}{2} (B_j f - B_j \tilde{f})^2])^{1/2} \\
&\quad \leqslant (E_i \otimes \tilde{E}_i\, \tfrac{1}{2} (f - \tilde{f})^2)^{1/2} + (E_i \otimes \tilde{E}_i (B_j f - B_j \tilde{f})^2)^{1/2} \\
&\quad \leqslant (\tfrac{1}{2} E_i \otimes \tilde{E}_i (f - \tilde{f})^2)^{1/2} + 2(E_i\, |B_j f|^2)^{1/2}.
\end{aligned} \tag{4.40}$$

Since E_i satisfies log-S with a constant $0 < c_0 < \infty$ (independent of external conditions $\sigma_{\Gamma\setminus i}$ and $i \in \Gamma$) so do $E_i \otimes \tilde{E}_i$ with the same constant. Therefore we have the mass gap inequality [23–25]

$$E_i \otimes \tilde{E}_i (f - \tilde{f})^2 \leqslant c_0 E_i \otimes \tilde{E}_i (|B_i f|^2 + |B_i \tilde{f}|^2) = 2c_0 E_i |B_i f|^2. \tag{4.41}$$

This together with (4.39) and (4.40) gives the following bound on the second term from the rhs of (4.27)

$$\begin{aligned} &\left| \frac{1}{2} \int_{-1}^{1} ds_j \, E_{i,s_j}(f^2_{s_j}, \partial_j U_{i,s_j}) \right| \\ &\quad \leqslant 2A_j (E_i f^2)^{1/2} \cdot 2^{-1/2} e^{8 \|B_j U_i\|_\infty} \|\bar{B}_j U_i - \bar{B}_j \tilde{U}_i\|_\infty \\ &\quad\quad \times (c_0^{1/2} (E_i |B_i f|^2)^{1/2} + 2(E_i |B_j f|^2)^{1/2}). \end{aligned} \tag{4.42}$$

Adding (4.42) to (4.35) and using property (4.21) we obtain

$$|B_j (E_i f^2)^{1/2}| \leqslant \alpha (E_i |B_j f|^2)^{1/2} + C_{ji} (E_i |B_i f|^2)^{1/2} \tag{4.43}$$

with

$$1 \leqslant \alpha \leqslant 2^{1/2} e^{6 \|B_j U_i\|_\infty} (1 + 2^{1/2} e^{2 \|B_j U_i\|_\infty} \|\bar{B}_j U_i - \bar{B}_j \tilde{U}_i\|_\infty) \tag{4.44}$$

and

$$0 < C_{ji} \leqslant 2^{-1/2} e^{8 \|B_j U_i\|_\infty} \|\bar{B}_j U_i - \bar{B}_j \tilde{U}_i\|_\infty c_0^{1/2}. \tag{4.45}$$

We may use the fact that

$$\|B_j U_i\|_\infty \leqslant \|\Phi\| \tag{4.46}$$

and

$$\|\bar{B}_j U_i - \bar{B}_j \tilde{U}_i\|_\infty \leqslant 2 \|B_j U_i\|_\infty \tag{4.47}$$

to get the simplified expressions (4.17) and (4.18). This ends the proof of Proposition 4.2. ∎

We see that the condition (4.16) given in Proposition 4.2 is of the same structure as the condition (0.12). Therefore if we would have $\alpha = 1$, we could repeat almost literally the considerations of Section 2 and get the same result as in Theorem 0.1. If $\alpha > 1$, then we still can get a nontrivial result imposing a stronger smallness condition on γ. We illustrate that by considering the following special case with $\Gamma = \mathbf{Z}^d$ and a local specification defined with a finite range interaction (i.e., $\Phi_X \equiv 0$ if $\operatorname{diam}(X) > r$ for some $r \in \mathbf{N}$). Let $\eta \equiv \alpha^{r^d}$.

THEOREM 4.3. *Let $\mathscr{E} = \{E_\Lambda^{\cdot}\}_{\Lambda \in \mathscr{F}}$ be a local specification on the space $\Omega \equiv \{-1, +1\}^{\Gamma}$ defined by a Gibbsian potential Φ of finite range r. Let the measure E_i^σ satisfy the* log-*Sobolev inequality with a constant $0 < c_0 < \infty$ independent of $i \in \Gamma$ and $\omega \in \Omega$. Suppose also that for any strictly positive function $f \in \Sigma$ and all i, $j \in \Gamma$, $i \neq j$*

$$|B_j(E_i f^2)^{1/2}| \leqslant \alpha(E_i |B_j f|^2)^{1/2} + C_{ji}(|E_i B_i f|^2)^{1/2} \tag{4.48}$$

with a constant $1 \leqslant \alpha < \infty$ independent of i and j, and a matrix $\mathbf{c} \equiv \{C_{ij} > 0\}_{i, j \in \Gamma}$ for which the correspondng parameter

$$\gamma \equiv \sup_{i \in \Gamma} \max \left(\sum_{j \in \Gamma} C_{ji}, \sum_{j \in \Gamma} C_{ij} \right) \tag{4.49}$$

fulfills

$$0 < \gamma\eta < 1. \tag{4.50}$$

Then the unique Gibbs measure $\mu \in \mathscr{G}(\mathscr{E})$ satisfies the log-*Sobolev inequality with coefficient*

$$0 < c \leqslant c_0 \eta^2 (1 - \gamma\eta)^{-2}. \tag{4.51}$$

Proof. The idea of the proof is the same as that of the corresponding result for continuous spin systems. Now, due to the constant $1 < \alpha < \infty$ in (4.48), we shall have to change the rhs of estimate (1.14) and multiply the first factor on its rhs by α. Expanding similarly the first term from the rhs of (1.14) we will gain another factor α. Continuing this procedure further we see that, because of finite range interaction, we may get at most a factor η before we get again a small factor C_{ji}. Taking this fact into account, a simple modification of the considerations from Section 1 yields the results of Theorem 4.3. ∎

Traditionally in statistical mechanics for a given interaction Φ one defines a family $\mathscr{E}_\beta$, $\beta > 0$, of local specifications at corresponding temperature β^{-1}, by the energy functional multiplied by factor β. Then from Proposition 4.2 we see that the condition (4.50) can be always satisfied by taking β sufficiently small. As a consequence of Theorem 4.3 we obtain that the corresponding stochastic dynamic is hypercontractive in the high temperature region. (For further investigation of discrete spin systems see [35].)

5. CONCLUDING REMARKS

The purpose of this paper was to consider the idea of [29], that a unique Gibbs measure $\mu \in \mathscr{G}(\mathscr{E})$ which can be represented in the form

$$\mu = \lim_{n \to \infty} E_{i_n} \cdots E_{i_1}, \tag{5.1}$$

with $\{E_{i_n}^{\cdot} \in \mathscr{E}\}_{n \in \mathbf{N}}$ and a sequence $\{i_n \in \mathbf{\Gamma}\}_{n \in \mathbf{N}}$ going infinitely many times through each point of the lattice, satisfies the logarithmic Sobolev inequality, provided it is fulfilled by the conditional measures E_i^{ω} with a constant independent of a point $i \in \mathbf{\Gamma}$ and external condition $\omega \in \Omega$. In other words (as stated in [29]) we wanted to show that log-Sobolev inequalities satisfied by conditional measures E_i^{ω} with the constants uniformly bounded in $\omega \in \Omega$ and $i \in \mathbf{\Gamma}$, and Dobrushin uniqueness theorem (assuring representation (1.5)), imply log-Sobolev inequalities for the unique Gibbs measure on the infinite lattice $\mathbf{\Gamma}$. This property of log-Sobolev inequalities for product measures is given in [1].

We have given a general condition for the above statement to be true and verified it in important cases of continuous and discrete spin systems on an infinite lattice.

By this we obtained some understanding why the measures on infinite dimensional spaces can satisfy log-Sobolev inequalities.

Let us mention that our method applies to the situations when the Bakry–Emery criterion does not work, like, e.g., discrete spin systems. (Although on the domain of intersection of both methods, Bakry–Emery works in a very nice and more effective way.)

In the work presented here we did not look for the optimal bounds of quantities appearing in our approach and therefore we did not show that our results hold in the full region of Dobrushin uniqueness (as formulated in [7–9]). Since this seemed to be less important, we postponed that to future investigations. (It is clear that from our condition (C) one can get an estimate of Dobrushin's interaction matrix.) However, the results of this and our work [35] (where we have shown log-S for Gibbs measures of one dimensional discrete spin systems with short range interactions at any temperature) suggest that log-Sobolev inequalities remain valid far outside Dobrushin's uniqueness region. That is in the region where instead we have its generalization, so-called Dobrushin-Schlosman uniqueness condition [36], satisfied. Let us note that this last condition implies that the unique Gibbs measure admits a representation

$$\mu = \lim_{n \to \infty} E_{\Lambda_{i_n}} \cdots E_{\Lambda_{i_1}} \tag{5.2}$$

with $\{\Lambda_{i_n} \in \mathscr{F}\}_{n \in \mathbf{N}}$ being a suitable sequence of sets of finite (fixed) volume, "covering" each point of lattice infinitely many times.

Let us now come back to the example following Proposition 3.4, concerning the Ising models in field (Kac–Siegert) representation. We note that when applying the Bakry–Emery criterion to a field measure one gets log-Sobolev inequalities in the uniqueness region of [32], which is also the Dobrushin uniqueness region. (At the same region we have also Brascamp–Lieb inequalities [37], which were the main tool used in [32].) We want to remark that one can use the mentioned criterion also for nonabsolutely summable interactions considered in [32], where the Dobrushin uniqueness method is noneffective. This result implies fast return to equilibrium (with an "evolution" modeled by stochastic dynamics) in the region where the phase transition is absent. It would be very interesting to recover this result directly for the corresponding discrete spin system and establish a correspondence between dynamics of field and spin systems connected by the Kac–Siegert transform. (For equilibrium physics such a correspondence has been established in [31].) In connection to what we said up to now a natural question arises: Does uniqueness of the Gibbs measure imply the log-Sobolev inequality? A physical intuition suggests that in general the answer should be no. It would be interesting to show that in particular the Kosterlitz–Thouless phase transition [38] is accompanied by dynamical phase transition, i.e., a change from exponential to algebraic rate of return to equilibrium. One may also expect to have a change to algebraic rate at the point of second order phase transition.

It would be also very interesting to show that spin glass phase transition [39] is accompanied by dynamical phase transition. (Note that such phenomena are experimentally observed. Let us also mention that in fact at present we have no rigorous results on the spin glass phase transition (but see [40]). One may expect to have such a phase transition for the long range interactions considered in [32, [31].) The other interesting problem concerns verification of log-S for probability measures on the space of real distributions $\mathscr{S}'(\mathbf{R}^d)$ constructed in euclidean field theory (see [41, 42] and references given there). Up to now we know that this inequality is satisfied for the free euclidean field defined by a Gaussian measure μ_0 on $\mathscr{S}'(\mathbf{R}^d)$, see [1], and the measures corresponding to the exponential model [34] in two dimensions (which can be proven, e.g., by lattice approximation and, e.g., the Bakry–Emery criterion). Other examples are provided by the time zero measure of the model with exponential interaction for which the global Markov property has been proven in [43] (see also [44, 45]) and where the physical Hamiltonian equals the closure of the Dirichlet form and has a mass gap in its spectrum. It would be also interesting to show log-S for euclidean measures of models with trigonometric and polynomial interactions and corresponding time zero measures.

The last interesting prblem we want to mention is the question of whether one can generalize the hypercontractivity theory based on the log-Sobolev inequality (as in [1]) to noncommutative setting, where we shall have to deal with a (Markov) semigroup $P_t \equiv e^{-tD}$, the generator D being defined in the simplest case by a quantum Dirichlet form of type

$$\langle A, B\rangle \equiv \omega(\delta(A^*)\,\delta(B))$$

given by a KMS-state ω on the $\mathscr{C}^*$-algebra (with unit), corresponding to derivation δ. The first step in this direction has been done in [46] (see also [47]). The other very interesting examples of quantum (dissipative) semigroups and operators one may find in [47, 48] (see also references therein). The investigations in this domain would be interesting for eventual application to noncommutative geometry and quantum spin systems on a lattice.

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